HYDRODYNAMIC POTENTIALS FOR THE MICROPOLAR NAVIER-STOKES PROBLEM

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An integral representation of linear and angular velocities and pressure for the description of linear stationary flows of micropolar viscous liquid media is obtained, and on its basis hydrodynamic potentials for the micropolar Navier-Stokes problem are introduced.

The linear stationary Navier-Stokes problem is reduced to the solution of the following system of differential equations [1]:

$$(\mu + \alpha)\,\Delta \mathbf{v} + (\mu + \lambda - \alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{v} + 2\alpha\,\mathrm{rot}\,\boldsymbol{\Omega} - \mathrm{grad}\,p = \rho \mathbf{f}\,,$$
$$(\nu + \beta)\,\Delta\boldsymbol{\Omega} + (\varepsilon + \nu - \beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\Omega} + 2\alpha\,\mathrm{rot}\,\mathbf{v} - 4\,\alpha\,\boldsymbol{\Omega} = \rho \mathbf{m}\,, \quad -\,\mathrm{div}\,\mathbf{v} = 0\,, \tag{1}$$

or in matrix form:

$$A\left(\frac{\partial}{\partial x}\right) V = F\,. \tag{2}$$

Here V is a column vector formed from the components of the vectors $\mathbf{v}$, $\boldsymbol{\Omega}$ and p; F is a column vector formed from $\rho\mathbf{f}$, $\rho\mathbf{m}$, and zero. The explicit form of the matrix $A(\partial/\partial x)$ is readily accessible from (1).

The Lagrange conjugate to the system of equations (1) is as follows:

$$(\mu + \alpha)\,\Delta \mathbf{u} + (\mu + \lambda - \alpha)\,\mathrm{grad}\,\mathrm{div}\,\mathbf{u} - 2\alpha\,\mathrm{rot}\,\boldsymbol{\omega} - \mathrm{grad}\,q = 0\,,$$
$$(\nu + \beta)\,\Delta\boldsymbol{\omega} + (\varepsilon + \nu - \beta)\,\mathrm{grad}\,\mathrm{div}\,\boldsymbol{\omega} - 2\alpha\,\mathrm{rot}\,\mathbf{u} - 4\,\alpha\,\boldsymbol{\omega} = 0\,, \quad -\,\mathrm{div}\,\mathbf{u} = 0\,. \tag{3}$$

We write it in matrix form as

$$A^*\left(\frac{\partial}{\partial x}\right) U = 0\,, \tag{4}$$

where U is a column vector formed from the components of the vectors $\mathbf{u}$, ω and q. System (1) differs from that presented in [1] by the notation for the constants. Namely, for symmetry in writing the equations in (1) the following notation is introduced:

$$\alpha = -\gamma\,, \quad \varepsilon = 2\tau\,, \quad \nu = \theta + \eta\,, \quad \beta = \theta - \eta\,, \tag{5}$$

where λ, μ, γ, η, τ, and θ are material parameters of the liquid medium [1].

For the operators $A(\partial/\partial x)$ and $A^*(\partial/\partial x)$ introduced here the following relationship holds:

$$U' A\left(\frac{\partial}{\partial x}\right) V - V' A^*\left(\frac{\partial}{\partial x}\right) U = \sum_{i=1}^{3} \partial_i\,(R_i - R_i^*)\,, \tag{6}$$

where

$$U' = (u_1\,,\ u_2\,,\ u_3\,,\ \omega_1\,,\ \omega_2\,,\ \omega_3\,,\ q)\,, \quad V' = (v_1\,,\ v_2\,,\ v_3\,,\ \Omega_1\,,\ \Omega_2\,,\ \Omega_3\,,\ p)\,, \quad \partial_i = \frac{\partial}{\partial x_i}\,, \quad i = \overline{1\,,\ 3}\,.$$

$$R_i = \sum_{j=1}^{3}\,[u_j \sigma_{ij} + \omega_j \mu_{ij}\,]\,, \quad R_i^* = \sum_{j=1}^{3}\,[v_j \sigma_{ij}^* + \Omega_j \mu_{ij}^*\,]\,,$$

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$$\sigma_{ij} = (-p + \lambda \operatorname{div} \mathbf{v}) \delta_{ij} + (\mu + \alpha) \partial_i v_j + (\mu - \alpha) \partial_j v_i - 2\alpha \sum_{k=1}^{3} \varepsilon_{ijk} \Omega_k ,$$

$$\mu_{ij} = \varepsilon \operatorname{div} \boldsymbol{\Omega} \delta_{ij} + (\nu + \beta) \partial_i \Omega_j + (\nu - \beta) \partial_j \Omega_i , \tag{7}$$

$$\sigma^*_{ij} = (q + \lambda \operatorname{div} \mathbf{u}) \delta_{ij} + (\mu + \alpha) \partial_i u_j + (\mu - \alpha) \partial_j u_i + 2\alpha \sum_{k=1}^{3} \varepsilon_{ijk} \omega_k ,$$

$$\mu^*_{ij} = \varepsilon \operatorname{div} \boldsymbol{\omega} \delta_{ij} + (\nu + \beta) \partial_i \omega_j + (\nu - \beta) \partial_j \omega_i .$$

From this the second Green formula follows for system (4):

$$\int_D \left[U' A \left(\frac{\partial}{\partial x} \right) V - V' A^* \left(\frac{\partial}{\partial x} \right) U \right] dv = \int_S \sum_{j=1}^{3} \left[u_j \sigma_{nj} + \omega_j \mu_{nj} - (v_j \sigma^*_{nj} + \Omega_j \mu^*_{nj}) \right] dS , \tag{8}$$

where D is the region in three-dimensional Euclidean space bounded by the surface S; $\mathbf{n}(n_1, n_2, n_3)$ is the outer normal to S; σ_{nj}, μ_{nj} and σ^*_{nj}, μ^*_{nj} are determined by the formulas

$$\sigma_{nj} = \sum_{i=1}^{3} \sigma_{ji} n_i , \quad \mu_{nj} = \sum_{i=1}^{3} \mu_{ji} n_i ; \quad \sigma^*_{nj} = \sum_{i=1}^{3} \sigma^*_{ji} n_i , \quad \mu^*_{nj} = \sum_{i=1}^{3} \mu^*_{ji} n_i . \tag{9}$$

We rewrite formula (8) in a different form by introducing the matrix of boundary stresses $B(\partial/\partial x, \mathbf{n})$:

$$B\left(\frac{\partial}{\partial x}, \mathbf{n} \right) = \| B_{ij} \|_{i,j=\overline{1,7}} , \quad B^*\left(\frac{\partial}{\partial x}, \mathbf{n} \right) = \| B^*_{ij} \|_{i,j=\overline{1,7}} , \tag{10}$$

where

$$B_{7k} = B^*_{7k} = 0 , \quad k = \overline{1, 7} ;$$

$$B_{ij} = (\mu + \alpha) \frac{\partial}{\partial n} \delta_{ij} + (\mu - \alpha) n_j \partial_i ; \quad B_{i+3,j+3} = (\nu + \beta) \frac{\partial}{\partial n} \delta_{ij} + \varepsilon n_i \partial_j + (\nu - \beta) n_j \partial_i ;$$

$$B_{i,j+3} = \sum_{k=1}^{3} 2 \alpha n_k \varepsilon_{jik} ; \quad B_{i7} = - n_i ; \quad B_{i+3,j} = B_{i+3,7} = 0 ; \tag{11}$$

$$B^*_{ij} = B_{ij} ; \quad B^*_{i+3,j+3} = B_{i+3,j+3} ; \quad B^*_{i,j+3} = - B_{i,j+3} ; \quad B^*_{i7} = - B_{i7} ;$$

$$B^*_{i+3,j} = B^*_{i+3,7} = 0 , \quad i, j = \overline{1, 3} .$$

With account for (10) formula (8) takes the following form:

$$\int_D \left[U' A \left(\frac{\partial}{\partial x} \right) V - V' A^* \left(\frac{\partial}{\partial x} \right) U \right] dv = \int_S \left[U' B \left(\frac{\partial}{\partial x}, \mathbf{n} \right) V - V' B^* \left(\frac{\partial}{\partial x}, \mathbf{n} \right) U \right] dS . \tag{12}$$

Using the Levi method [2] the matrix of fundamental solutions of system (1) can be constructed. Omitting intermediate calculations we present its final form:

$$\Gamma(x) = \| \Gamma_{ij} \|_{i,j=\overline{1,7}} , \tag{13}$$

where

$$\Gamma_{ij} = \frac{1}{4\pi\mu} \left[\frac{1}{r} - \frac{\alpha}{\mu + \alpha} \frac{\exp(-\sigma_1 r)}{r} \right] \delta_{ij} + \frac{1}{8\pi\mu} \partial_i \partial_j \left[\frac{\nu + \beta}{2\mu} \frac{\exp(-\sigma_1 r) - 1}{r} - r \right] ;$$

$$\Gamma_{i+3,j+3} = \frac{1}{4\pi(\nu + \beta)} \frac{\exp(-\sigma_1 r)}{r} \delta_{ij} + \frac{1}{16\pi} \partial_i \partial_j \left[\frac{\exp(-\sigma_2 r) - \exp(\sigma_1 r)}{\alpha r} - \frac{\exp(-\sigma_1 r) - 1}{\mu r} \right] ; \tag{14}$$

$$\Gamma_{i+3,j} = \Gamma_{i,j+3} = -\frac{1}{8\pi\mu}\sum_{k=1}^{3}\varepsilon_{ijk}\,\partial_k\left[\frac{\exp(-\sigma_1 r) - 1}{r}\right];$$

$$\Gamma_{7j} = \Gamma_{j7} = -\frac{1}{4\pi}\,\partial_j\frac{1}{r};$$

$$\Gamma_{7,j+3} = \Gamma_{j+3,7} = 0; \quad \sigma_1^2 = \frac{4\alpha\mu}{(\mu+\alpha)(\nu+\beta)}; \quad \sigma_2^2 = \frac{4\alpha}{(\varepsilon+2\nu)}; \quad r = \sqrt{\left(\sum_{i=1}^{3} x_i^2\right)}; \quad \partial_i = \frac{\partial}{\partial x_i};$$

$$\Gamma_{77} = -(\mu+\alpha)\,\delta(x), \quad i,\ i = \overline{1,3}.$$

The singular part of matrix (13) is as follows:

$$\Gamma_0(x) = \|\Gamma_{ij}^0\|_{i,j=\overline{1,7}}, \tag{15}$$

where

$$\Gamma_{ij}^0 = \frac{1}{8\pi(\mu+\alpha)}\left[\frac{1}{r}\delta_{ij} + \frac{x_i x_j}{r^3}\right];$$

$$\Gamma_{i+3,j+3}^0 = \frac{1}{8\pi(\nu+\beta)(\varepsilon+2\nu)}\left[\frac{\beta+\varepsilon+3\nu}{r}\delta_{ij} + (\varepsilon-\beta+\nu)\frac{x_i x_j}{r^3}\right];$$

$$\Gamma_{j,7}^0 = \Gamma_{7,j}^0 = \frac{1}{4\pi}\frac{x_j}{r^3}; \quad \Gamma_{i+3,j}^0 = \Gamma_{i,j+3}^0 = 0; \quad \Gamma_{77}^0 = -(\mu+\alpha)\,\delta(x), \quad i,\ j = \overline{1,3}. \tag{16}$$

Direct computations show that at $x \neq y$ the columns of the matrix $\Gamma(x - y)$ defined by the expressions (13) and (14) are the solutions of the homogeneous system of equations (1) with respect to the variable x, whereas with respect to the variable y the columns are the solutions of the homogeneous conjugate system (3). It follows from expressions (16) that Γ_{ij}^0 and Γ_{j7}^0, $i, j = \overline{1,3}$ represent the tensor of singular solutions for the following linearized Navier-Stokes system:

$$(\mu+\alpha)\,\Delta \mathrm{v} - \mathrm{grad}\,p = \rho \mathrm{f},$$

At the same time, $\Gamma_{i+3,\,j+3}^0$, $i, j = \overline{1,3}$ is the tensor of fundamental (singular) solutions of the system of equations

$$\mathrm{div}\,\mathrm{v} = 0.$$

and therefore, based on the usual considerations, we obtain from formula (12) the following integral representation of the regular solutions of system (1) within the region D [3, 4]:

$$(\nu+\beta)\,\Delta\boldsymbol{\Omega} + (\varepsilon+\nu-\beta)\,\mathrm{grad\,div}\,\boldsymbol{\Omega} = \rho \mathrm{m},$$

$$a(x)\,V(x) = \int_S \left\{\Gamma'(x-y)\,B\left(\frac{\partial}{\partial y}, \mathrm{n}(y)\right)V(y) - \left[V'(y)\,B^*\left(\frac{\partial}{\partial y}, \mathrm{n}(y)\right)\Gamma(x-y)\right]\right\} d_yS + \int_D \Gamma(x-y)\,F(y)\,d_yv, \tag{17}$$

where

$$a(x) = \begin{cases} 1, & x \in D, \\ \frac{1}{2}, & x \in S, \\ 0, & x \,\overline{\in}\, D. \end{cases}$$

Explicit integral representations for all components of the linear and angular velocity and the pressure can be obtained from formula (17) using ordinary matrix multiplication procedures. This formula provides a basis for introducing the hydrodynamic micropolar potentials of the bulk, single, and double layers (the surface integrals and the volume integral in (17) are such; the latter should be considered to be a volume hydrodynamic micropolar potential). In addition, formula (17) makes it possible to write out the boundary conditions for a numerical-analytical solution of the basic boundary-value problems for system (1) when either the components of the vectors of linear and angular velocity or the moment and force stresses or various combinations of them are specified on the boundary of the region.

In conclusion, it should be noted that formula (17) contains as a special case well-known results of the classical Navier-Stokes problem [4, 5].

NOTATION

σ_{ij}, μ_{ij}, components of tensors of force and moment stresses; $\mathbf{v}(v_1, v_2, v_3)$, $\boldsymbol{\Omega}(\Omega_1, \Omega_2, \Omega_3)$, vectors of linear and angular velocity; p, pressure; ρ, density; $\mathbf{f}(f_1, f_2, f_3)$, $\mathbf{m}(m_1, m_2, m_3)$, spatially distributed forces and moments; λ, μ, η, τ, θ, γ, material parameters of the liquid medium (coefficients of volume, shear, and rotational viscosity, and a measure of the bonding of a liquid particle with its surroundings); δ_{ij}, Kronecker symbol; ε_{ijk}, Levi-Civita symbol; $\delta(x-y)$, Dirac delta function; $x(x_1, x_2, x_3)$, point of three-dimensional space; $r = (\sum_{i=1}^{3}(x_i - y_i)^2)^{1/2}$, distance between the points $x(x_1, x_2, x_3)$ and $y(y_1, y_2, y_3)$; $\Delta = \sum_{i=1}^{3} \partial_i^2$, Laplace operator; $\partial_i = \partial/\partial x_i$, $\partial_i^2 = \partial^2/\partial x_i^2$, $\partial_i \partial_j = \partial^2/\partial x_i \partial x_j$.

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